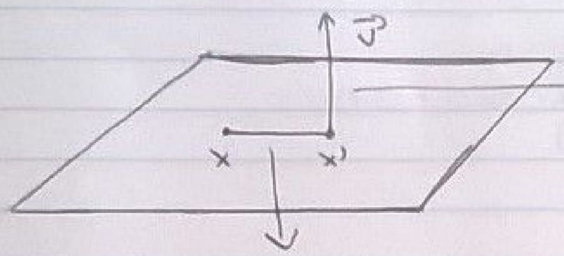
MAT B41 Week 2 Notes

1. Planes

- Def: A plane in $\mathbb{R}^n$ may be decided by a point on the plane and a vector, $\vec{v}$, that is orthogonal to the plane.

- In $\mathbb{R}^n$, let $x = (x_1, x_2, \dots x_n)$ be a point on the plane, $\vec{v} = [v_1, v_2, \dots v_n]$ be a vector orthogonal to the plane and $x' = (x_1', x_2', \dots x_n')$ be an arbitrary point on the plane.



$\vec{v}$ is orthogonal to the plane.

$x' - x$ is on the plane.

$$\vec{v} \perp (x' - x)$$
$$\Rightarrow \vec{v} \cdot (x' - x) = 0$$
$$\Rightarrow [v_1, v_2, \dots v_n] \cdot [x_1' - x_1, x_2' - x_2, \dots x_n' - x_n]$$
$$\Rightarrow v_1(x_1' - x_1) + \dots v_n(x_n' - x_n) = 0$$
$$\Rightarrow v_1 x_1' - v_1 x_1 + \dots + v_n x_n' - v_n x_n = 0$$
$$\Rightarrow v_1 x_1' + v_2 x_2' + \dots + v_n x_n' = \underbrace{v_1 x_1 + \dots + v_n x_n}_{\text{This is a constant. We'll denote this as D.}}$$
$$\Rightarrow \underbrace{v_1 x_1' + v_2 x_2' + \dots v_n x_n' = D}_{\text{This is the equation of a plane.}}$$

- E.g. 1 Find an equation of the plane that passes through the point $(1,1,-1)$ and is orthogonal to the line

$$\begin{cases} x = 1+t \\ y = 1-3t \\ z = -7t \end{cases}, \quad t \in \mathbb{R}$$

Soln:

The direction vector of the line is $[1,-3,-7]$.

$\vec{v} = [1,-3,-7]$

Let (x,y,z) be an arbitrary point on the plane.

$$x - 3y - 7z = d$$
$$d = (1)(1) + (-3)(1) + (-7)(-1)$$
$$= 1 - 3 + 7$$
$$= 5$$

$\therefore$ The eqn of the plane is $x - 3y - 7z = d$

- E.g. 2 Find the point of intersection between the plane $x - 3y - 7z$ and the line

$$\begin{cases} x = 2t \\ y = 1+2t \\ z = -2-t \end{cases}, \quad t \in \mathbb{R}$$

Soln: $x - 3y - 7z = 5$

$$(2t) - 3(1+2t) - 7(-2-t) = 5$$
$$2t - 3 - 6t + 14 + 7t = 5$$
$$3t = -6$$
$$t = -2$$

When $t = -2$, $x = -4$, $y = -3$, $z = 0$

$\therefore (-4,-3,0)$ is the point of intersection.

- 2 planes are parallel if their normal vectors are parallel.

- If 2 planes are not parallel, then they must intercept at a straight line and the angle between the 2 planes is the angle between their normal vectors.

- E.g. 3 Find the angle between each of the 2 planes.

a) $\begin{cases} -2x + 6y + 14z = 5 \\ x - 3y - 7z = 5 \end{cases}$

$$\vec{v_1} = [-2, 6, 14]$$
$$\vec{v_2} = [1, -3, -7]$$

$$\vec{v_1} = -2\vec{v_2}$$

$\therefore$ The 2 planes are parallel.

b) $\begin{cases} 2x + 3y - z = 4 \\ x - 3y - 7z = 5 \end{cases}$

$$\vec{v_1} = [2, 3, -1]$$
$$\vec{v_2} = [1, -3, -7]$$

$$\theta = \arccos\left(\frac{\vec{v_1} \cdot \vec{v_2}}{\|\vec{v_1}\| \, \|\vec{v_2}\|}\right)$$

$$= \arccos\left(\frac{2 - 9 + 7}{\sqrt{14} \cdot \sqrt{59}}\right)$$

$$= \arccos(0)$$
$$= \frac{\pi}{2}$$

$\therefore$ $\vec{v_1}$ and $\vec{v_2}$ are orthogonal.

2. Curves:

- For instance, a curve may be represented by a path of a particle. In R^2, curves may not be represented by functions.

- Circles:

$\leftarrow y = \sqrt{r^2 - x^2}$

$\leftarrow y = -\sqrt{r^2 - x^2}$

A circle has a formula: $x^2 + y^2 = r^2$.
Note: A circle is not a function, but can be represented by 2 functions, $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$

Here are some parametric equations for a circle:

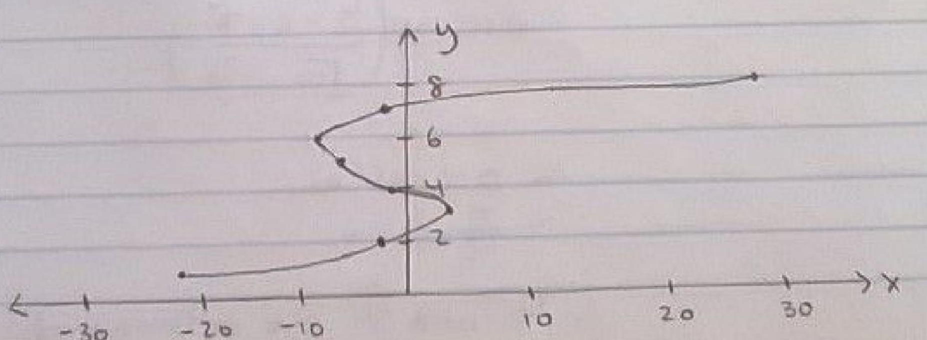
1. $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \; 0 \le \theta \le 2\pi$

2. $\begin{cases} x = r\sin\theta \\ y = r\cos\theta \end{cases}, \; 0 \le \theta \le 2\pi$

- E.g. 3 Sketch the curve $x = t^3 - 4t^2 + 2$, $y = t + 3$, $-2 \le t \le 5$

Soln:

t	-2	-1	0	1	2	3	4	5
x	-22	-3	2	-1	-6	-7	2	27
y	1	2	3	4	5	6	7	8



- E.g. 4 Find the parametric equations that represent the elliptic curve

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

Solution:

$$\frac{(x-x_0)^2}{a^2} = \left(\frac{x-x_0}{a}\right)^2$$

let $\cos\theta = \frac{x-x_0}{a} \rightarrow x = x_0 + a\cos\theta$

$$\frac{(y-y_0)^2}{b^2} = \left(\frac{y-y_0}{b}\right)^2$$

let $\sin\theta = \frac{y-y_0}{b} \rightarrow y = y_0 + b\sin\theta$

We are letting $\cos\theta = \frac{x-x_0}{a}$ and $\sin\theta = \frac{y-y_0}{a}$

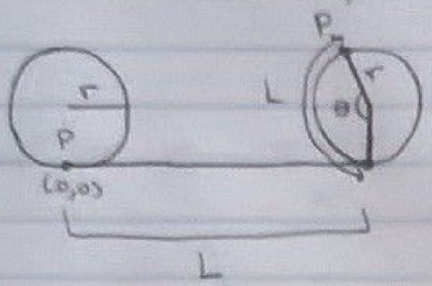
because we know that $\sin^2\theta + \cos^2\theta = 1$.

$$\begin{cases} x = x_0 + a\cos\theta \\ y = y_0 + b\cos\theta \end{cases}, \quad 0 \le \theta \le 2\pi$$

is the parametric equation.

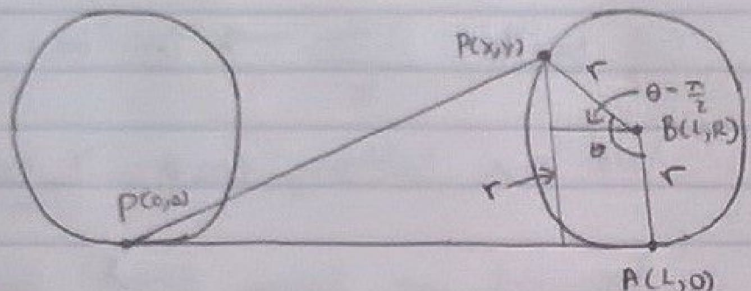
- E.g. 5 The curve traced at a point, P, on the circumference of a circle as the circle rolls along a straight line is a cycloid. Let the circle have a radius of r and roll along the line. Suppose the position of P starts at the origin. Find the parametric equation of the cycloid.

Soln:



The length of the arc from the point on the line to P is the length that the circle has rolled.

$L = \theta R$ (Equation of an arc)



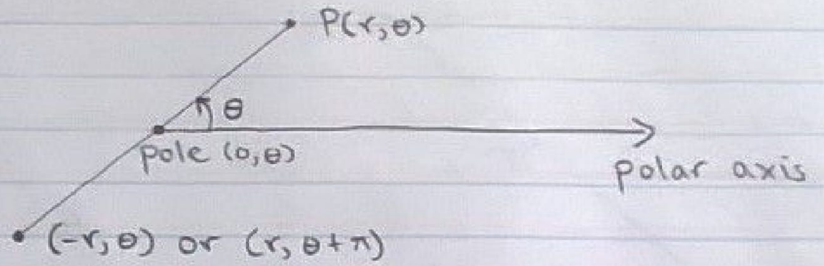
$$x = L - r\cos(\theta - \tfrac{\pi}{2})$$
$$y = r + r\sin(\theta - \tfrac{\pi}{2})$$

By using the trig double angle formulas, we get

$$\begin{cases} x = L - r\sin\theta \\ y = r + r\cos\theta \end{cases}$$

3. Polar Coordinates:

- Another way of representing where points are.
- The origin is called the pole.
- The axes are called polar axes.



- In the cartesion coordinate system, we used (x,y) to represent a point's location in R^2. In the polar coordinate system, we use (r,θ) instead. r is the magnitude of the line. θ is the angle between the line and the horizontal polar axis.

- If θ moves counterclockwise, then it is positive. If θ moves clockwise, then it is negative.

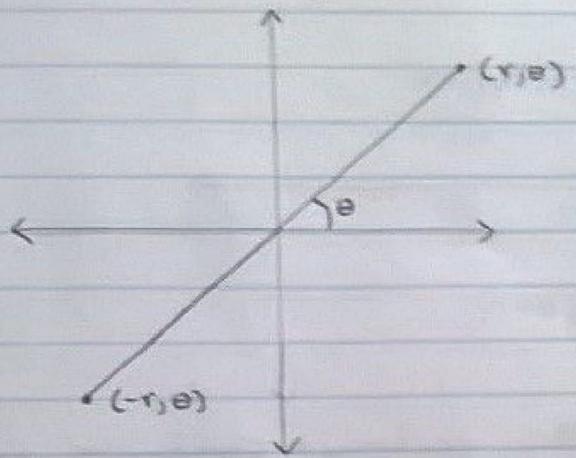
- The pole can be represented as $(0,\theta)$.

- In polar coordinates, we allow r to be negative. Furthermore, $(-r,\theta)$ and (r,θ) lies on the same line and that line must go through the pole.
  $(-r,\theta) = (r, \theta+\pi)$

- You can use the following equations to change between polar and cartesian coordinates
  1. $x = r\cos\theta$
     $y = r\sin\theta$
  2. $r = \sqrt{x^2+y^2}$
     $\theta = \arctan\left(\frac{y}{x}\right)$, if $x>0$ and $y>0$

- If $r>0$, (r,θ) lies in the same quadrant as θ.

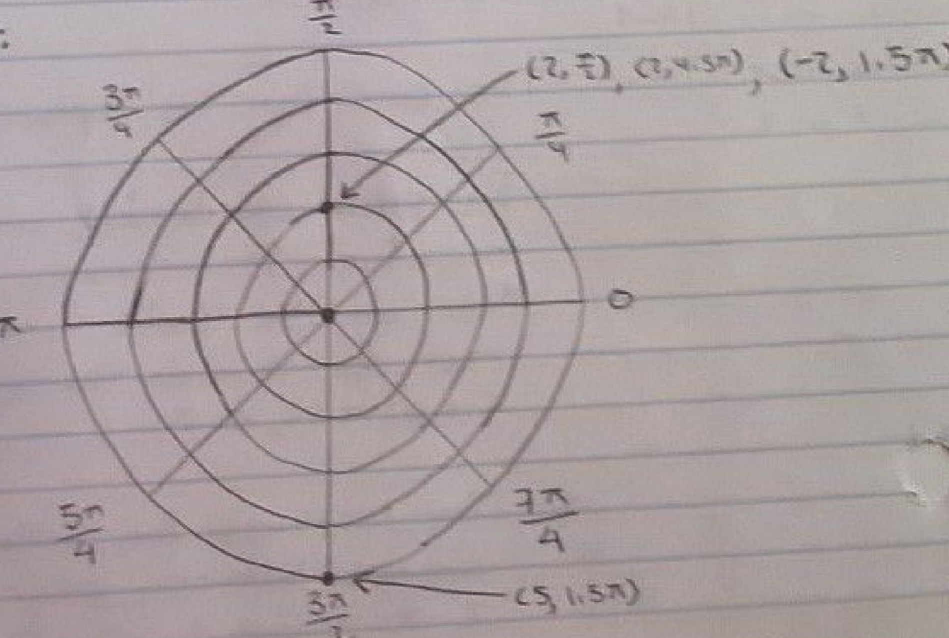
- If $r<0$, (r,θ) lies in the quadrant opposite of θ.



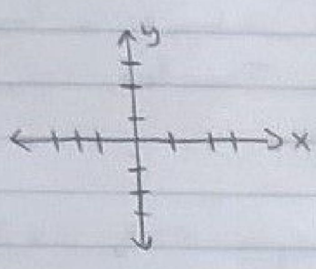
From this picture, you see that if $r>0$, (r,θ) lies in the same quadrant as θ and if $r<0$, (r,θ) lies in the quadrant opposite of θ.

- E.g. 6 Plot the points $(2, \frac{\pi}{2})$, $(5, 1.5\pi)$, $(-2, 1.5\pi)$ and $(2, 4.5\pi)$.

Soln:



In the cartesian coordinate system, we used to draw this



The rings of a polar coordinate system are analogous to the markings on the cartesian plane. In our each, each ring goes up by 1 unit.
I.e. The pole has a modulus of 0. The first ring has a modulus of 1. The second ring has a modulus of 2. And so on.

To point the points, find which ring your r value corresponds to and then find the angle that your θ value corresponds to.

In the case of $(2, 4.5\pi)$, because $4.5 > 2$, that means the angle has made a full trip around the circle already. Therefore, to get a θ value between 0 and 2π, just subtract 2 from 4.5 until you get to a number between 0 and 2, inclusive.

$4.5 - 2 = 2.5$, $2.5 > 2$ so we need to subtract again

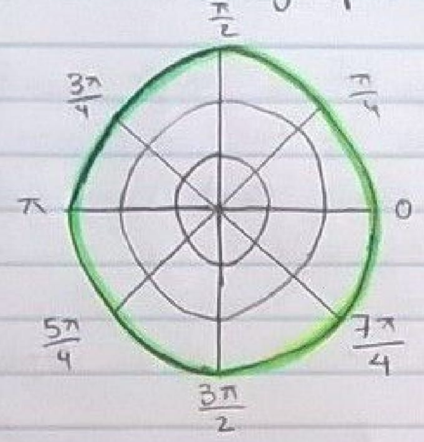
$2.5 - 2 = 0.5$, $0 \leq 0.5 \leq 2$

Therefore, $(2, 4.5\pi)$ is at the same place as $(2, 0.5\pi)$ on the plane.

In the case of $(-2, 1.5\pi)$, we can rewrite it as $(2, 1.5\pi + \pi)$, which is equal to $(2, 2.5\pi)$. Since $2.5 > 2$, we need to subtract 2 from it. This gives us $(2, 0.5\pi)$. Therefore, $(-2, 1.5\pi)$ is at $(2, 0.5\pi)$ on the plane.

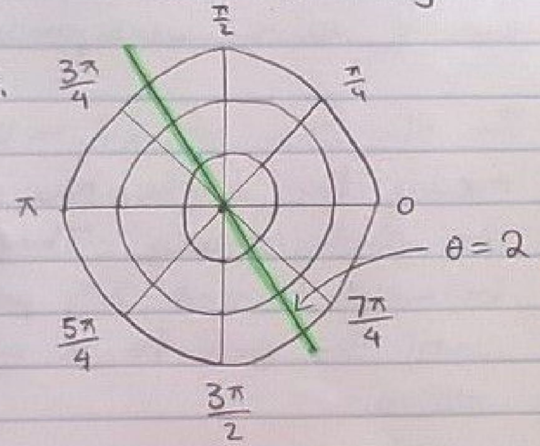
- E.g. 7 Sketch the graph $r=3$, $\theta=2$, and $r=\theta$

Soln: 1.



The coloured ring is $r=3$.

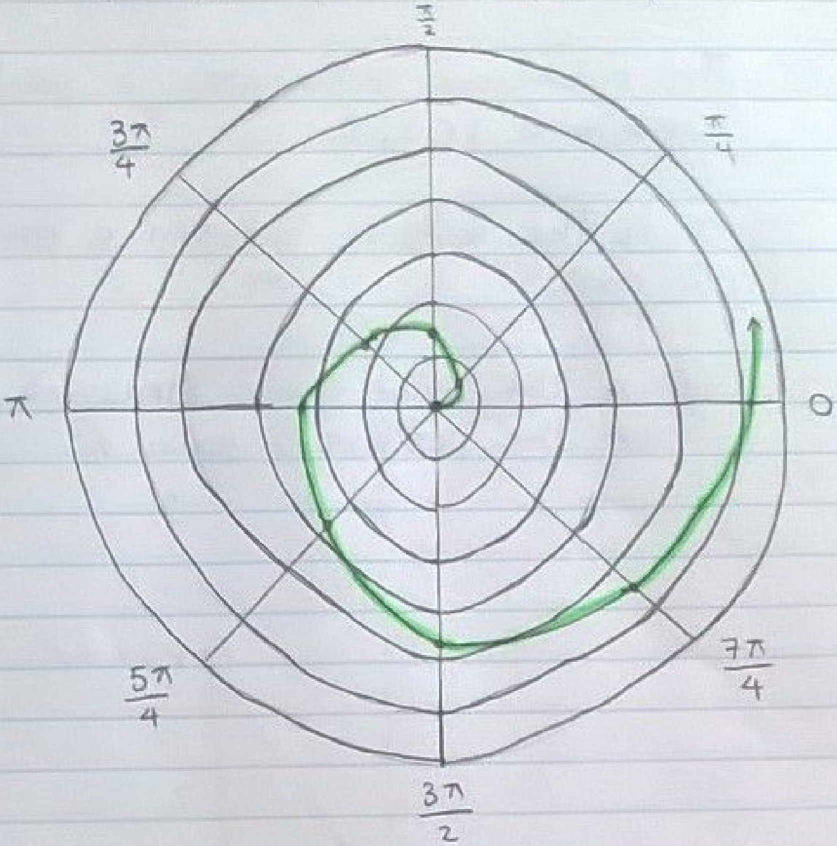
2.



We know that $\frac{\pi}{2} \approx 1.57$ and $\frac{3\pi}{4} \approx \frac{9}{4} \approx 2.25$.
Therefore, $\theta = 2$ is a line between $\frac{\pi}{2}$ and $\frac{3\pi}{4}$.

3. For $r = \theta$, we need to make a chart first.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	0	0.79	1.57	2.36	3.14	3.93	4.71	5.50	6.28



The green line is $r = \theta$.

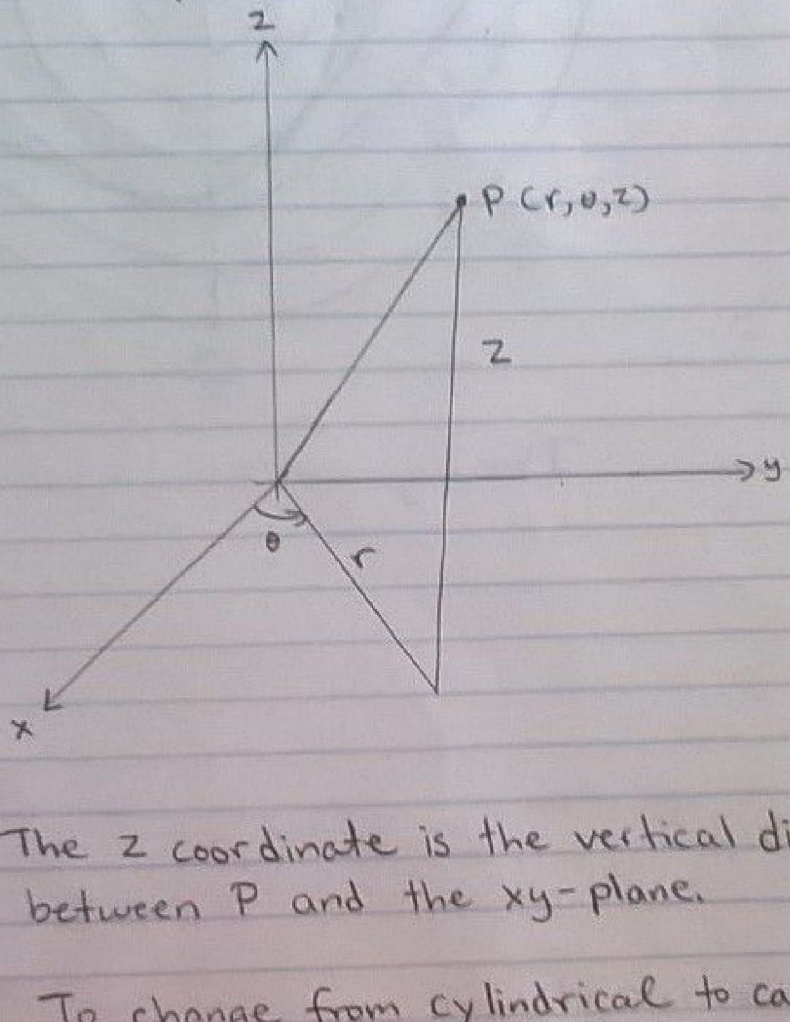
4 Cylindrical Coordinates:

- When we extend polar coordinates from R^2 to R^3, the result is cylindrical coordinates.

- In cylindrical coordinates, a point has coordinates (r, θ, z).

- r is the distance between a point and the z-axis.

- θ is the polar angle measured counterclockwise from the positive x-axis.



- The z coordinate is the vertical distance between P and the xy-plane.

- To change from cylindrical to cartesian

$x = r\cos\theta$

$y = r\sin\theta$

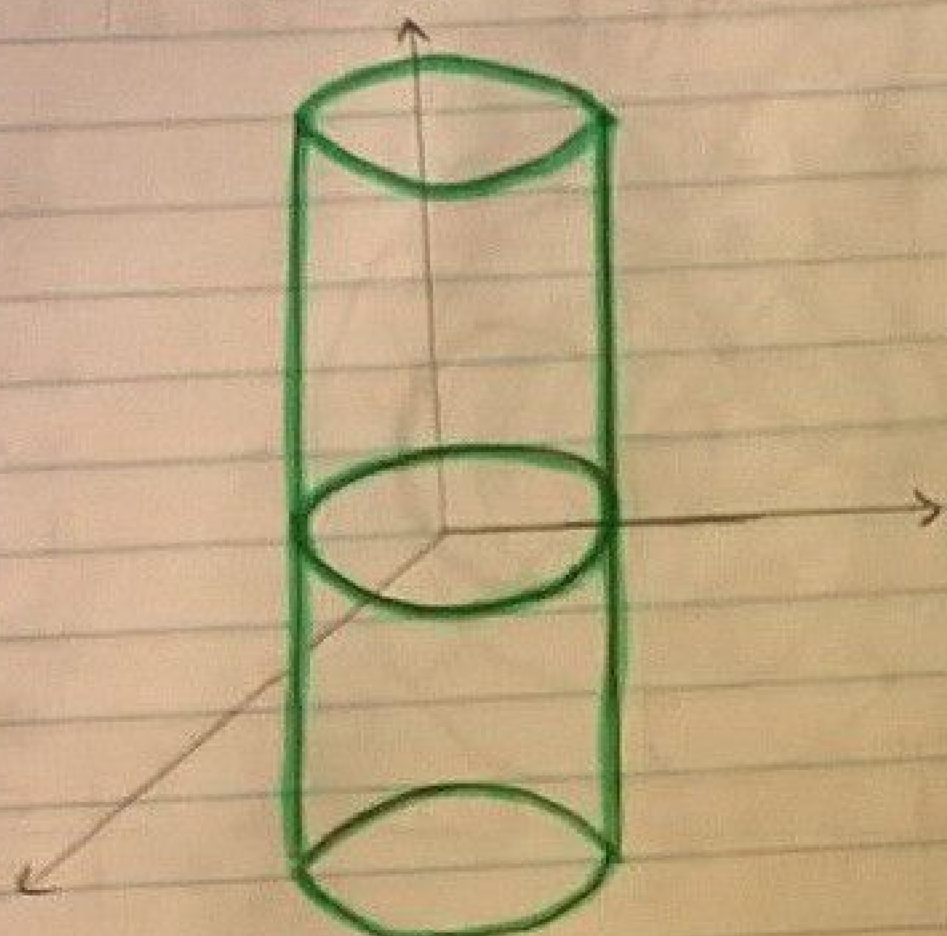
$z = z$

- E.g. 8 Sketch the graphs of
  1. $r=3$
  2. $\theta=2$

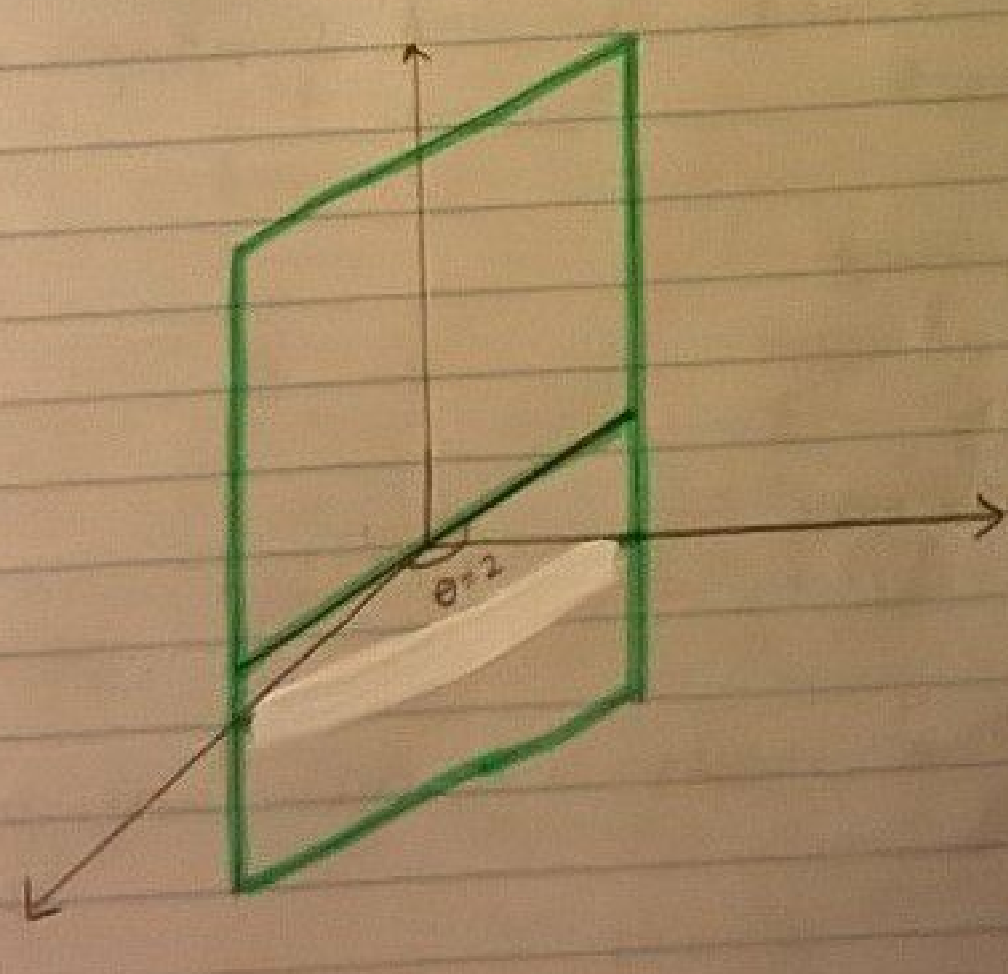
  in cylindrical coordinates.

Soln:

1. Recall that $r=3$ is a circle in polar coordinates.



2. Recall that $\theta=2$ is a line in polar coordinates.

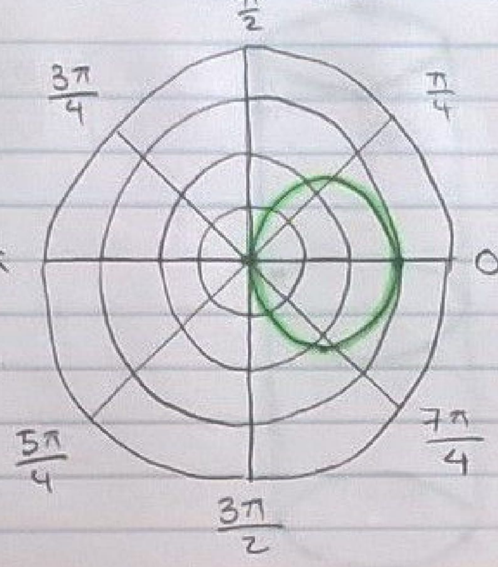


- E.g. 9 Sketch the graphs

1. $r = 3\cos\theta$
2. $r = \sin(2\theta)$

1. $r = 3\cos\theta$

Since $-1 \le \cos\theta \le 1$, $-3 \le r \le 3$
However, since $(-r, \theta) = (r, \theta + \pi)$, $0 \le r \le 3$.



When $\theta = 0$, $r = 3$
when $\theta = \frac{\pi}{2}$, $r = 0$
when $\theta = \pi$, $r = -3$, but by using the $(-r, \theta) = (r, \theta + \pi)$ conversion, we get $(3, 2\pi)$.

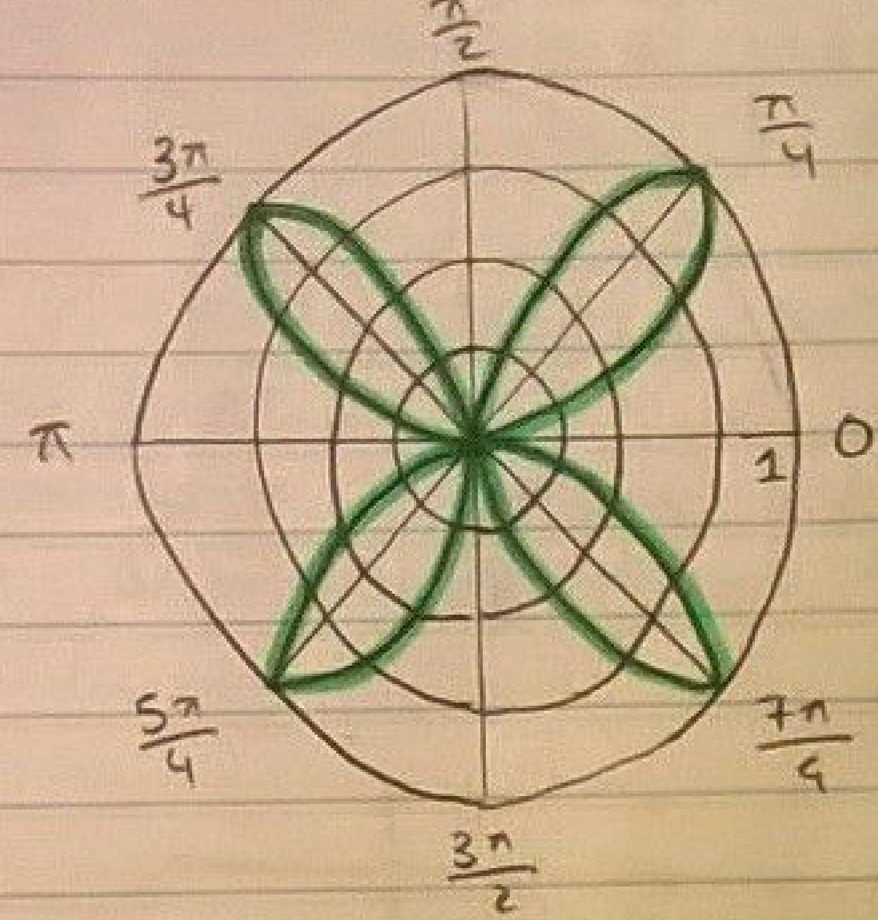
We get a circle with a radius of $\frac{3}{2}$.
We can use the conversion formula between polar coordinates and cartesian coordinates to check.

$$r = 3\cos\theta$$
$$\to r^2 = 3r\cos\theta$$
$$\to x^2 + y^2 = 3x$$
$$\to x^2 - 3x + y^2 = 0$$
$$\to \left(x - \tfrac{3}{2}\right)^2 + y^2 = \left(\tfrac{3}{2}\right)^2$$

$\therefore$ The result is a circle with a radius of $\frac{3}{2}$.

2. $r = \sin(2\theta)$

Since $-1 \le \sin(2\theta) \le 1$, $-1 \le r \le 1$, but using the formula $(-r, \theta) = (r, \theta + \frac{\pi}{2})$, we get $0 \le r \le 1$.



θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	0	$\sqrt{3}/2$	1	$\sqrt{3}/2$	1	-1	0	1	0	-1	0

E.g. 10 Eliminate the parameters to find a Cartesian equation of the curve.

$$\begin{cases} x = t^2 + 1 \\ y = t + 1 \end{cases}$$

Solution.

$t = y - 1$ OR $t = \pm\sqrt{x-1}$

$x = (y-1)^2 + 1$ $\quad$ $y = \pm\sqrt{x-1} + 1$

Either equation is fine.